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Complexity results for bi-criteria cyclic scheduling problems*

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Abstract
The minimization of the amount of initial tokens in a Timed Event Graph (in short TEG) under throughput constraint is a crucial problem in industrial area such as the design of manufacturing systems or embedded systems. The aim of this paper is to prove that this problem is strongly related to the K-colorability of a graph. Its NP-completeness and complexity results for other particular cyclic scheduling problems are then derived.

Keywords : Timed Event Graphs, Throughput, Work-In-Process, Complexity.

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1. Introduction

Cyclic scheduling problems, in which a set of generic tasks have to be performed infinitely often, have numerous practical applications. In these systems, the throughput is usually an important performance measure for designers (for a survey, see [1]).
In the context of manufacturing Systems, Timed Event Graphs (in short TEG) are widely used to model complex assembly lines. The main feature of this subclass of Petri Nets is that each place p has exactly one input transition and one output transition. Hence, TEG are conflict free. Workshop (resp. products) are usually modelled by transitions (resp. tokens). Between two succesive transformations, products (i.e. tokens) have to be stored or have to be moved from a workshop to an other one. The amount of products, also called Work In Process (WIP in short), that have to be stored or moved may have economical consequences. Therefore, the main problem of designer is to devise an initial configuration of WIP that allows the system to reach a given productivity and that uses the smallest amount of WIP. Many optimization algorithms and heuristics were developped in order to solved it and some variants (see. [2-6]).
Some embedded applications may also be modelled using Timed Event Graphs: the synchronous Data-flow [7] formalism, which includes TEG, is usually considered for these applications. In this case, transitions represent processes and places represent intermediate buffers. Tokens model data exchanged between processes. Because of the high cost of the memories, the size of the intermediate buffers must be minimum. This criteria can be expressed by associating a backwards place $p' = (t_j, t_i)$ to any place $p = (t_i, t_j)$ (see. [8]). The total size of the buffer corresponding to p is then the amount of tokens in places p and p'. The bi-criteria

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problem studied is then to minimize the total number of initial tokens under throughput constraint. Researchers have addressed some closely related problems by various approaches such as integer linear programming [9] or more recently by model checking [10].
This paper is dedicated to the study of the complexity of this bi-criteria problem for a TEG. Despite numerous heuristics and exact algorithms developed previously to solve it, there is no complexity results. We exhibit in this paper a central bi-criteria problem and we prove the relationship with the K-colorability problem. NP-completeness and some complexity results for other particular cyclic scheduling problems are then derived.
This paper is structured as follows: Section 2 is dedicated to basic definitions and the description of the main decision problem. In Section 3, we prove that it is equivalent to the K-colorability problem and we deduce its NP-completeness. Other results concerning important open decision problems are derived from the former result in Section 4. We conclude lastly with some perpectives in Section 5.

2. Basic definitions

We suppose that the reader is aware with theoretical background of Petri Net (see. [11] for further details). However, this section recalls the main definitions concerning this paper.
Let us consider a Timed Event Graph $\mathcal{G} = (P, T, l)$ given by a set of places $P = \{p_1, \cdots, p_m\}$, a set of transitions $T = \{t_1, \cdots, t_n\}$ and a function $l : T \rightarrow \mathbb{R}$ such that, for any $t \in T$, $l(t)$ is the duration of a firing of t. Every place $p \in P$ is defined between two transitions t_i and t_j and is denoted by $p = (t_i, t_j)$ (see Figure 1 here under). We will assume that there is at most one place

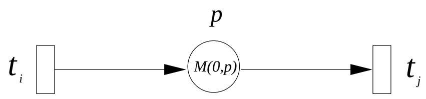

Figure 1: A place $p = (t_i, t_j)$.

$p = (t_i, t_j)$ defined from t_i to t_j. Moreover, we suppose that two successive firings of a same transition cannot overlap: this is modelled by a self loop place $p = (t_i, t_i)$, $\forall t_i \in T$. For a sake of simplicity, these loops are not presented in figures.
For any transition $t \in T$, we set:

$$\mathcal{P}^+(t) = \{p = (t, t') \in P, t' \in T\} \qquad \text{and} \qquad \mathcal{P}^-(t) = \{p = (t', t) \in P, t' \in T\}$$

The incidence matrix I associated with a TEG $\mathcal{G}$ is defined by $|P| \times |T|$ values in $\{-1, 0, 1\}$ such that, for any couple $(p, t) \in P \times T$:

- $I[p, t] = 1$ (resp. -1) if $p \in \mathcal{P}^+(t)$ (resp. if $p \in \mathcal{P}^-(t)$) ;
- $I[p, t] = 0$ otherwise.

A P-semiflow is a vector $Y \in \mathbb{N}^{|P|}$ such that ${}^TY.I = 0$.
A path μ of $\mathcal{G}$ is defined as a sequence of α places such that $\mu = (p_1 = (t_1, t_2), p_2 = (t_2, t_3), \ldots,$ $p_\alpha = (t_\alpha, t_{\alpha+1}))$. If this path is closed (i.e. $t_1 = t_{\alpha+1}$), then μ is a circuit.
We denote by $M(\tau, p)$ the instantaneous marking of the place p at time instant $\tau \geq 0$. The marking $M(0, p)$ is called the initial marking of place p and $M(\mathcal{G})$ points out the initial marking

of the TEG $\mathcal{G}$. The semantic rules of TEG are the same as Petri Nets. This proposition defines special structures of TEG:

Proposition 2.1. *Let $\mathcal{G}$ be a* TEG*:*

*i. $\mathcal{G}$ is a Symmetric Timed Event Graph (*STEG *in short) if each place $p = (t_i, t_j) \in P$ is associated with a backward place $p' = (t_j, t_i) \in P$. (p, p') is then called a couple of backward places.*

ii. An initial marking $M(\mathcal{G})$ of a STEG $\mathcal{G}$ is said minimally bounded if there is exactly one token for each couple of backward place. More formally, for any couple (p, p') of backward places,

$$M(0, p) + M(0, p') = 1$$

A TEG $\mathcal{G}$ with an initial marking is live if each transition $t \in T$ may be fired infinitely often. Setting $H(\mathcal{C}) = \sum_{p \in P \cap \mathcal{C}} M(0, p)$ the length of a circuit $\mathcal{C}$ of $\mathcal{G}$, it is proved in [12] that M is a live marking iff the height of every circuit of $\mathcal{G}$ is strictly positive.
The throughput is usually considered as a classical performance criterion for TEG. If $\sigma(t, n)$, with $t \in T$ and $n \in \mathbb{N}$ denotes the starting time of the nth firing of t, it is defined as

$$\lim_{n \to \infty} \min_{t \in t} \frac{n}{\sigma(t, n)}$$

Setting the lenght of a circuit $\mathcal{C}$ to $L(\mathcal{C}) = \sum_{t \in T \cap \mathcal{C}} l(t)$, Chrétienne proved the following theorem:

Theorem 2.1. *[13] Let $\mathcal{G}$ be a TEG with a given live initial marking $M(\mathcal{G})$. The maximum throughput of $\mathcal{G}$ is equal to*

$$\lambda(M(\mathcal{G})) = \min_{\mathcal{C} \in \mathcal{C}_{\mathcal{G}}} \frac{H(\mathcal{C})}{L(\mathcal{C})}$$

where $\mathcal{G}_G$ denotes the set of circuits of $\mathcal{G}$.

A critical circuit c of $\mathcal{G}$ is such that $\frac{H(c)}{L(c)} = \lambda(M(\mathcal{G}))$. Using Koenigs lemma (see. [14]), Chrétienne also proved the following theorem:

Theorem 2.2. *[13] Let $\mathcal{G}$ be a TEG with a live initial marking $M(\mathcal{G})$. Then, every critical circuit may be decomposed into elementary critical circuits.*

This last theorem allows us to consider only elementary critical circuits.
The aim of this paper is to study the complexity of the decision problem defined as follows: MAX THROUGHPUT - MIN BOUNDED:

Instance: A STEG $\mathcal{G} = (T, P, l)$ such that $l(t) = 1, \forall t \in T$ and an integer $K > 1$.

Question: is there an initial live marking $M(\mathcal{G})$ minimally bounded such that $\lambda(M(\mathcal{G})) \geqslant \frac{1}{K}$?

3. Complexity of MAX THROUGHPUT - MIN BOUNDED

The aim of this section is to prove that MAX THROUGHPUT - MIN BOUNDED is NP-complete using a reduction from the FLOW RATIO problem. In [15], Minty describes this problem and exhibits its relationship with the graph colorability problem.
Let $G = (V, E)$ be a non-oriented simple graph. An edge orientation is a function $o : E \to V \times V$ such that, for any $e = \{x, y\} \in E$, $o(e) \in \{(x, y), (y, x)\}$.

The flow ratio $\rho(o(G))$ of an orientation o is then defined as follows: for any cycle c of G, we may associate two integers $n_c(o)$ and $m_c(o)$ corresponding respectively to the number of edges of c in a way and in its opposite. Then,

$$\rho(o(G)) = \max_{c \in \mathcal{C}_G} \left(\frac{n_c(o)}{m_c(o)}, \frac{m_c(o)}{n_c(o)} \right)$$

FLOW RATIO is then defined as follows:

Instance: A non oriented graph $G = (V, E)$, an integer $L > 0$.

Question: Is there an orientation o of G such that $\rho(o(G)) \leq L$?

We define a transformation f from an instance of FLOW RATIO to an instance of MAX THROUGHPUT - MIN BOUNDED. Let I be an instance of FLOW RATIO given by a graph $G = (V, E)$ and an integer L. The corresponding instance $f(I)$ of MAX THROUGHPUT - MIN BOUNDED is defined as:

1. Any vertex $x \in V$ is associated with a transition $t_x \in T$ with $l(t_x) = 1$;
2. Any edge $e = (x, y) \in T$ corresponds to two places $p_1 = (t_x, t_y)$ and $p_2 = (t_y, t_x)$;
3. $K = L + 1$.

Now, if o is an orientation of G, a minimally bounded initial marking for $\mathcal{G}$ can be deduced by setting

1. $M(0, p) \in \{0, 1\}$ for any $p \in P$;
2. For any place $p = (t_x, t_y)$, $M(0, p) = 1$ iff $o(\{x, y\}) = (x, y)$.

Notice that f is bijective. Moreover, any minimally bounded initial marking of $\mathcal{G}$ is associated with exactly one orientation of G. We are now ready to prove the relationship between $\rho(o(G))$ and $\lambda(M(\mathcal{G}))$.

Lemma 3.1. *For any orientation o of G associated with a minimally bounded initial marking M of $\mathcal{G}$,*

$$\lambda(M(\mathcal{G})) = \frac{1}{\rho(o(G)) + 1}$$

Proof. Let c be a cycle of G. Then, c is associated with two circuits of $\mathcal{G}$ in opposite directions denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$ (see. figure 2 on the next page).
We can suppose without loss of generality that the direction of $\mathcal{C}_1$ corresponds to the $n_c(o)$ arcs of c. So, $H(\mathcal{C}_1) = n_c(o)$ and $H(\mathcal{C}_2) = m_c(o)$. Moreover, $L(\mathcal{C}_1) = L(\mathcal{C}_2) = n_c(o) + m_c(o)$.
The throughput $\lambda_{\mathcal{C}}$ of the STEG composed only by $\mathcal{C}_1$ and $\mathcal{C}_2$ is:

$$\begin{aligned} \lambda_{\mathcal{C}} &= \min\left(\tfrac{n_c(o)}{n_c(o)+m_c(o)}, \tfrac{m_c(o)}{n_c(o)+m_c(o)}\right) \\ &= \frac{1}{\max\left(\frac{n_c(o)+m_c(o)}{n_c(o)}, \frac{n_c(o)+m_c(o)}{n_c(o)}\right)} \\ &= \frac{1}{\max\left(\frac{m_c(o)}{n_c(o)}, \frac{n_c(o)}{m_c(o)}\right)+1} \end{aligned}$$

So, if ρ_c denotes the flow ratio of c, we get $\lambda_{\mathcal{C}} = \frac{1}{\rho_c+1}$. Now, if c is a circuit with a maximum flow-ratio in G, one of the corresponding circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ have a minimum throughput in $\mathcal{G}$, so the equality is true. $\square$

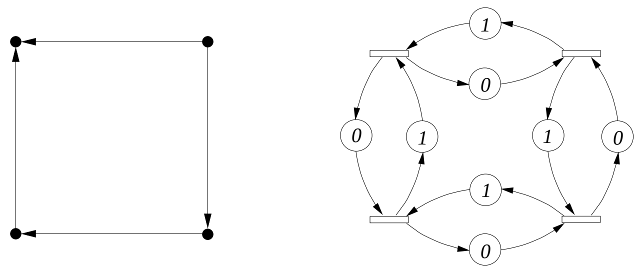

Figure 2: A cycle c is depicted on the left hand side of the figure. The associated marked STEG is given on the right hand side.

Lemma 3.2. f *is a polynomial transformation.*

Proof. f may be computed polynomially. The correctness of the transformation may be easily deduced from lemma 3.1. ☐

As the result described in [15] was established before the NP-COMPLETENESS theory advent, we clearly show that FLOW RATIO is intractable.

Theorem 3.1. FLOW RATIO *is* NP-COMPLETE.

Proof. FLOW RATIO belongs to NP since the flow ratio of a solution may be computed efficiently from the throughput of the associated marked STEG (computed by Karp's algorithm [16] for example). Moreover, Minty's lemma [15] states that a graph $G = (V, E)$ is K-colorable if and only if there exists an edge orientation of E such that the flow ratio is lower or equal to $K - 1$. This result can be seen as a polynomial reduction of GRAPH K-COLORABILITY to FLOW RATIO. As GRAPH K-COLORABILITY is NP-COMPLETE (see. [17]), the theorem holds. ☐

We deduce now our complexity result:

Theorem 3.2. MAX THROUGHPUT - MIN BOUNDED *is* NP-COMPLETE.

Proof. MAX THROUGHPUT - MIN BOUNDED belongs to NP since the throughput can be efficiently computed by using Karp's algorithm [16] and one can polynomially check that the marking properties are met.
Since f is bijective, f and f^{-1} define a polynomial reduction between FLOW RATIO and MAX THROUGHPUT - MIN BOUNDED. By theorem 3.1, the proof is completed. In fact, these two problems are equivalent. ☐

4. Complexity results for two closely related problems

In this part, we set computational results for two very close problems. The first one, called MARKING OPTIMIZATION was studied by several authors [3,5], but its complexity was up to now unknown. The second one is a subproblem of MAX THROUGHPUT - MIN BOUNDED.

4.1. The MARKING OPTIMIZATION problem

MARKING OPTIMIZATION problem was first introduced by Laftit and Proth [3].
MARKING OPTIMIZATION:

Instance: $\mathcal{G} = (P, T, l)$ is a TEG , Y is a P-semiflow of $\mathcal{G}$, $N \in \mathbb{N}^\star$ and $Q \in \mathbb{N}^\star$.

Question: Is there an initial marking $M(\mathcal{G})$ such that ${}^{T}Y \cdot M(0,p) \leqslant N$ and $\lambda(M(\mathcal{G})) \geqslant \frac{1}{Q}$?

We can derive from theorem 3.2 the following theorem:

Theorem 4.1. *The* MARKING OPTIMIZATION *problem is* NP-COMPLETE.

Proof. MARKING OPTIMIZATION is in NP.
We prove now that MAX THROUGHPUT - MIN BOUNDED is a sub-problem of MARKING OPTIMIZATION. Let I be an instance of MAX THROUGHPUT - MIN BOUNDED given by a STEG $\mathcal{G}$ and an integer K. Since $\mathcal{G}$ is a STEG, $|\mathcal{P}^{+}(t)| = |\mathcal{P}^{-}(t)|, \ \forall t \in T$. Hence, each column of the incidence matrix of the graph has exactly as many positive entries as negative ones. Therefore, the unit vector $1^{|P|}$ is a P-semiflow. Moreover, by setting $N = \frac{|P|}{2}$, the constraint ${}^{T}Y \cdot M(0,p) \leqslant N$ is equivalent to require that M should be minimally bounded. Lastly, setting $Q = K$ we deduce that I is an instance of MARKING OPTIMIZATION. □

4.2. The MAX INTRINSIC THROUGHPUT problem
Let $\mathcal{G} = (P, T, l)$ be a STEG. We study now the minimization of the total number of initial tokens in such a way that the TEG structure has no more influence on the throughput. In this case, the critical circuit is then a self-loop on a transition t_i with a maximum duration.
MAX INTRINSIC THROUGHPUT:

Instance: $\mathcal{G} = (P, T, l)$ is a STEG and $Z \in \mathbb{N}^{\star}$.

Question: Is there an initial live marking $M(\mathcal{G})$ such that $\sum_{p \in P} M(0,p) \leqslant Z$ and such that $\lambda(M(\mathcal{G})) \geqslant \frac{1}{\max\limits_{t_j \in T}(l(t_j))}$?

In [2,4,5,6], authors have studied the problem where the TEG is not restricted to be symmetric. However, the forthcoming NP-completeness result holds even if this assumption is removed.

Theorem 4.2. MAX INTRINSIC THROUGHPUT *is polynomial for* $l(t) = 1, \forall t \in T$.

Proof. We prove that the initial marking $M(0,p) = 1, \ \forall p \in P$ is a solution. Indeed, since $l(t_i) = 1, \ \ \forall t_i \in T$, then $\frac{1}{\max\limits_{t_j \in T}(l(t_j))} = 1$ and $\lambda(M(\mathcal{G})) \geq 1$. As $\mathcal{G}$ is symmetric, every couple of backward places needs at least two tokens to reach the throughput 1. We deduce that $|P|$ is a lower bound of the total initial number of tokens.
So, if $|P| > Z$, there is no live marking M such that $\sum_{p \in P} M(0,p) \leqslant Z$. Otherwise, the initial marking built previously allows to answer the question positively. □

If the durations of the transitions firings are not required to be equal to 1, the previous algorithm builds an initial marking with an intrinsic maximum throughput. However, the total number of tokens is not necessarily minimum, as illustrated by Figure 3 on the facing page.

Theorem 4.3. MAX INTRINSIC THROUGHPUT *is* NP-COMPLETE.

Proof. The problem belongs to NP. We exhibit a reduction from MAX THROUGHPUT - MIN BOUNDED to MAX INTRINSIC THROUGHPUT. Let $\mathcal{G} = (P, T, l)$ be an instance of MAX THROUGHPUT - MIN BOUNDED defined by a STEG $\mathcal{G} = (P, T, l)$ and an integer value K. We build the corresponding instance of MAX INTRINSIC THROUGHPUT as follows:

1. We build another STEG $\mathcal{G}^{\star}$ by adding a transition $t^{\star}$ with $l(t^{\star}) = K$ and a couple of backward places $(t, t^{\star})$ and $(t^{\star}, t)$ for a fixed transition $t \in T$.

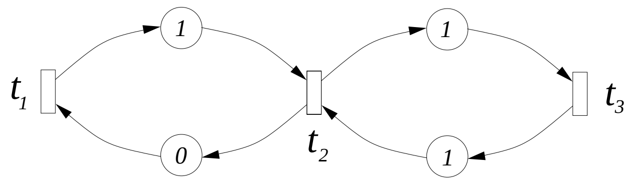

Figure 3: The STEG has three transitions with $l(t_1) = 1$, $l(t_2) = 2$ and $l(t_3) = 3$. The initial marking reaches the maximum intrinsic throughput of the system i.e. $\frac{1}{3}$.

2. We set $Z^\star = \frac{|P|}{2} + 2$. Notice that the maximum intrinsic throughput is $\frac{1}{K}$.

- If M is a solution for the instance of MAX THROUGHPUT - MIN BOUNDED, we build a solution $M^\star$ for the corresponding instance of MAX INTRINSIC THROUGHPUT by adding 1 token on each added places. The total number of initial tokens is $Z^\star = \frac{|P|}{2} + 2$. Now, if $c^\star$ is a critical circuit of $\mathcal{G}^\star$, then by theorem 2.2, $c^\star$ can be considered as elementary. We get then two cases:

  1. If $c^\star$ is included in $\mathcal{G}$, then $\lambda(M^\star(\mathcal{G}^\star)) \geq \frac{H(c^\star)}{L(c^\star)} \geq \frac{1}{K}$;
  2. Else, since $c^\star$ is elementary, it composed only by the two transitions t and $t^\star$ and thus
  $$\frac{H(c^\star)}{L(c^\star)} = \frac{2}{K+1} > \frac{1}{K}$$

  So, $M^\star$ is a solution to the corresponding instance of MAX INTRINSIC THROUGHPUT.

- Let us suppose now that $M^\star$ is a solution for the instance of MAX INTRINSIC THROUGHPUT. Setting $M(0,p) = M^\star(0,p)$ for every place $p \in P$, we prove that M is a solution to the corresponding instance of MAX THROUGHPUT - MIN BOUNDED.

  By definition of M and $M^\star$, $\lambda(M(\mathcal{G})) \geq \lambda(M^\star(\mathcal{G}^\star)) \geq \frac{1}{K}$.

  Now, let c be the circuit of $\mathcal{G}^\star$ composed by transitions t and $t^\star$. Since $M^\star$ is a live marking, $H(c) \geq 1$. So, $\frac{H(c)}{L(c)} = \frac{H(c)}{K+1} \geq \frac{1}{K}$. So, $H(c) \geq 1 + \frac{1}{K}$ and then c has at least two tokens.

  So, the number of initial tokens of $\mathcal{G}$ is upper bounded by $\frac{|P|}{2}$. Since $M^\star$ is live, M is live and then, for every couple (p, p') of backward places,
  $$M(0,p) + M(0,p') \geq 1$$

  So, there is exactly one token for every couple of backward places and M is minimally bounded.

□

5. Conclusions

We have presented in this paper several complexity results on TEG. We have exhibit a decision problem that seems to play a central role in the study of bi-criteria problem for TEG. It allows to set the complexity of important problem even for instance with severe constraints.

Moreover, we proved the equivalence between our bi-criteria scheduling problem and the K-colorabilty problem. We hope that this equivalence will lead to original algorithms for the resolution of our scheduling problem.

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