Complexity results for bi-criteria cyclic scheduling problems*

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Abstract

The minimization of the amount of initial tokens in a Weighted Timed Event Graph (in short WTEG) or a Timed Event Graph (in short TEG) under throughput constraint is a crucial problem in industrial area such as the design of manufacturing systems or embedded systems. Two important sub-problems are studied in this paper: the first one concerns the maximization of the throughput for minimum places capacities of a TEG. It is proved NP-complete by a polynomial reduction with the K-colorability problem. The second one is the minimization of the overall places capacities with a maximum throughput. This problem is also proved NP-complete for a TEG. A polynomial sub-case and a 2-approximation polynomial algorithm for a WTEG are then provided.

Keywords: Weighted Timed Event Graphs, Throughput, Buffer requirement, Complexity, Approximation algorithm.

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1 Introduction

Cyclic scheduling problems, in which a set of generic tasks have to be performed infinitely often, have numerous practical applications. In these systems, the throughput is usually an important performance measure for designers (for surveys, see [9, 16]).

In this paper, we consider that the constraints on tasks are modelled using Weighted Timed Event Graph (in short WTEG) $\mathcal{G}$ which is a subclass of Petri Nets. Transitions are associated with generic tasks and their firings have a given duration. Each place p has exactly one input and one output transition weighted by respective values $w(p)$ and $v(p)$: at the completion of a firing of the input transition of p, $w(p)$ tokens are added to p. At the firing of the output transition of p, $v(p)$ tokens are removed from p. If $v(p) = w(p) = 1$ for every place, $\mathcal{G}$ is a Timed Event Graph (in short TEG).

WTEG and TEG are widely used for modelling and solving practical cyclic scheduling problems. In the context of manufacturing systems, they are considered to model complex assembly lines. Workshop (*resp.* products) are usually modelled by transitions (*resp.* tokens). Between two successive transformations, products (*i.e.* tokens) have to be stored or have to be moved from a workshop to an other one. The amount of products, also called Work In Process (WIP in short), that have to be stored or moved may have economical consequences. Therefore, the main problem for designers is to devise an initial configuration of WIP that allows the system to reach a given productivity and that uses the smallest amount of WIP. Many optimization algorithms and heuristics were developed in order to solved it and some variants for (non weighted) Timed Event Graphs or Weighted Timed Event Graphs (*see.* [4, 6, 7, 12, 20]).

Synchronous Data-Flow [13] is a well-known formalism considered for modelling some embedded applications and is equivalent to WTEG. In this case, transitions represent processes and places intermediate buffers. Tokens model data exchanged between processes. The size of the intermediate buffers must be minimum because of the high cost of the memories. This criteria can be expressed by associating a backwards place $p' = (t_j, t_i)$ to any place $p = (t_i, t_j)$ (*see.* [15]). The total size of the buffer corresponding to p is then the amount of tokens in places p and p'. The bi-criteria problem studied is then to minimize the total number of initial tokens under throughput constraint. Researchers have addressed some closely related problems by various approaches such as integer linear programming [8] or more recently by model checking [21].

This paper is dedicated to the study of the complexity of this bi-criteria problem for a WTEG. Despite numerous heuristics and exact algorithms developed previously to solve it, there is no complexity results. We exhibit in this paper a central bi-criteria problem and we prove its relationship with the K-colorability problem. NP-completeness and some complexity results for other particular cyclic scheduling problems are then derived. A 2-approximation algorithm is lastly provided to solve the minimization of the overall places capacities to achieve a maximum throughput.

This paper is structured as follows: Section 2 is dedicated to basic definitions and the description of our main decision problems. Some important previous results are also recalled here. Section 3 is dedicated to the study of maximizing the throughput for minimally bounded places of a TEG. We prove that it is equivalent to the K-colorability problem and we deduce its NP-completeness. Section 4 considers the minimization of the overall places capacities with a maximum throughput for a TEG. It is proved that the decision problem is NP-complete, but polynomial for unit duration firings and a 2-approximation algorithm is considered. These two last results are extended to WTEG in Section 5. We conclude lastly with some perpectives in Section 6.

2 Notations and definitions

2.1 Basic definitions and example

We suppose that the reader is aware with theoretical background of Petri Nets (*see.* [19] for further details). However, this subsection recalls the main definitions concerning this paper.

2.1.1 Weighted Event Graph

A Weighted Event Graph $\mathcal{G} = (P, T)$ (in short WEG) is given by a set of places $P = \{p_1, \ldots, p_m\}$ and a set of transitions $T = \{t_1, \ldots, t_n\}$. Every place $p \in P$ is defined between two transitions t_i and t_j and is denoted by $p = (t_i, t_j)$. The arcs (t_i, p) and (p, t_j) are valued by strictly positive integers denoted respectively by $w(p)$ and $v(p)$ (see Figure 1). At each firing of the transition t_i (*resp.* t_j), $w(p)$ (*resp.* $v(p)$) tokens are added to (*resp.* removed from) place p. If $v(p) = w(p) = 1$ for every place $p \in P$, then $\mathcal{G}$ is an

Event Graph (in short EG). We will assume that there is at most one place

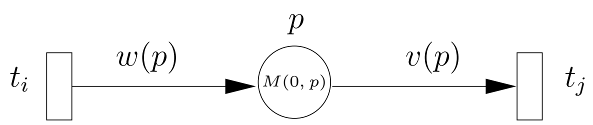

Figure 1: A place $p = (t_i, t_j)$ of a Generalized Timed Event Graph.

$p = (t_i, t_j)$ defined from t_i to t_j. For any transition $t \in T$, we set

$$\mathcal{P}^+(t) = \{p = (t, t') \in P, t' \in T\} \qquad \text{and} \qquad \mathcal{P}^-(t) = \{p = (t', t) \in P, t' \in T\}.$$

For any place $p \in P$, gcd_p and lcm_p denotes respectively the greatest common divisor and the least common multiple of integers $v(p)$ and $w(p)$.

The incidence matrix I associated with a WEG $\mathcal{G}$ is defined by $|P| \times |T|$ values in $\mathbb{Z}$ such that, for any couple $(p, t) \in P \times T$:

- $I[p, t] = w(p)$ (*resp.* $-v(p)$) if $p \in \mathcal{P}^+(t)$ (*resp.* if $p \in \mathcal{P}^-(t)$) ;
- $I[p, t] = 0$ otherwise.

A P-semiflow is a vector $Y \in \mathbb{R}^{|P|}$ such that ${}^{\intercal}Y.I = 0$.

A path μ of $\mathcal{G}$ is defined as a sequence of α places such that $\mu = (p_1 = (t_1, t_2), p_2 = (t_2, t_3), \ldots, p_\alpha = (t_\alpha, t_{\alpha+1}))$. If this path is closed (*i.e.* $t_1 = t_{\alpha+1}$), then μ is a circuit. An initial marking is a function $M_0 : P \to \mathbb{N}$ such that, for any $p \in P$, $M_0(p)$ denotes the initial number of tokens in the place p. A WEG is marked if it is associated with an initial marking.

2.1.2 Timed Weighted Event Graph

A Timed Weighted Event Graph (in short TWEG) is a WEG associated with a function $\ell : T \to \mathbb{R}^{+\star}$ such that, for any $t \in T$, $\ell(t)$ is the duration of a firing of t. It is usually denoted by $\mathcal{G} = (P, T, \ell)$. For every place $p = (t_i, t_j) \in P$, $w(p)$ tokens are added to p exactly $\ell(t_i)$ times units after the firing of t_i. $v(p)$ tokens are removed from p at the firing of t_j.

We suppose that two successive firings of the same transition cannot overlap: this is modelled by a self loop place $p = (t_i, t_i)$, $\forall t_i \in T$ with $w(p) = v(p) = 1$ and $M_0(p) = 1$. For a sake of simplicity, these loops are not presented in figures.

We denote by $M(\tau, p)$ the instantaneous marking of the place p at time instant $\tau \geq 0$. The marking $M(0, p)$ is called the initial marking of place p and $M(\mathcal{G})$ points out the initial marking of the WTEG $\mathcal{G}$. $\mathcal{G}$ is a marked WTEG (*resp.* TEG) if it is associated with an initial marking $M(\mathcal{G})$.

2.1.3 Example

A Digital Signal Processing (DSP in short) may be modelled by a TWEG: transitions represent processes that may be executed infinitely often. Buffers allow communications between processes. A buffer has exactly one input and one output and data stored in a buffer have the same size. At the beginning (*resp.* ending) of each iteration, a process has to read (*resp.* write) a fixed amount of data in each input (*resp.* output) buffers. Let us suppose that the WTEG pictured by Figure 2 depicts a DSP. Transition t_3 needs 2 data from t_1 and 7 data from t_2 to be executed once. At its completion, 5 data are are temporarily stored in place $p_3 = (t_3, t_4)$ to be sent to t_4.

The completion of one execution of each process t_i has a duration $\ell(t_i)$. Because of the cost of the memories in embedded systems, the capacity of the buffers (*i.e.* the maximum number of data that can be simultaneously stored) should be limited. So, the problem consists of finding an initial marking of the places and their capacities $C(p_i)$, $i \in \{1, \dots, m\}$ such that several criteria are realized:

- The system should be live: each transition $t \in T$ may be fired infinitely often.
- The surface of a buffer on the chip is usually proportional to the size of the data stored in it. More formally, if y_p, $p \in P$ denotes the size of one data stored in p, the criteria considered is the minimization of the global storage surface defined as $\sum\limits_{p \in P} y_p \cdot C(p)$.
- The throughput of the system should be maximum.

In the following, we recall some results concerning these three criteria.

2.2 Weight of paths and Liveness of a WEG

The complexity of the determination of the liveness of a marked WEG is unknown. Algorithms developed up to now to answer this question are pseudo-polynomial (*see.* as example [18]).

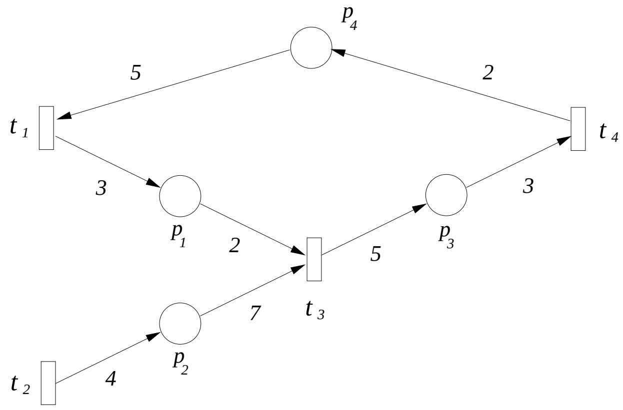

Figure 2: A Weighted Timed Event Graph with $\ell(t_1) = 5$, $\ell(t_2) = 2$, $\ell(t_3) = 6$ and $\ell(t_4) = 3$.

However, it is polynomial for marked Event Graphs. Indeed, setting $H(\mathcal{C}) = \sum_{p \in P \cap \mathcal{C}} M(0, p)$ the height of a circuit $\mathcal{C}$ of $\mathcal{G}$, it is proved in [3] that M is a live marking iff the height of every circuit of $\mathcal{G}$ is strictly positive.

In the case of Weighted Event Graph, the follwing simple necessary condition of liveness was noticed by several authors [10, 18, 22]: let us define the weight (or gain) of every path μ of a Weighted Event Graph $\mathcal{G}$, denoted by $W(\mu)$ as

$$W(\mu) = \prod_{p \in P \cap \mu} \frac{w(p)}{v(p)}.$$

Then, if a marked Weighted Event Graph $\mathcal{G}$ is live, every circuit has a weight equal to or greater than 1. This condition is trivially not sufficient: it is always fulfilled for Event Graphs and not sufficient to decide about the liveness.

2.3 Symmetric Weighted Event Graphs

One of the criteria considered here is the minimization of a linear function of the total capacity of the places. A bounded capacity $C(p)$ of a place $p = (t_i, t_j) \in P$ may be modelled by adding another place $p' = (t_j, t_i)$ with $w(p') = v(p)$, $v(p') = w(p)$ and such that the initial marking verifies $M_0(p) + M_0(p') = C(p)$. So, a Symmetric WEG defined as follows may be associated to any WEG with bounded capacity places:

Proposition 2.1. *Let $\mathcal{G}$ be a WEG:*

1. *$\mathcal{G}$ is a Symmetric Weighted Event Graph (SWEG in short) if each place $p = (t_i, t_j) \in P$ is associated with a backward place $p' = (t_j, t_i) \in P$ with $w(p') = v(p)$ and $v(p') = w(p)$. (p, p') is then called a couple of backward places.*

2. *An initial marking $M(\mathcal{G})$ of a SWEG $\mathcal{G}$ is said minimally bounded if for any couple (p, p') of backward places,*

$$M_0(p) + M_0(p') = M_{min}(p) = w(p) + v(p) - gcd_p.$$

Note that $M_{min}(p) = 1$, $\forall p \in P$ if $\mathcal{G}$ is a Symmetric Event Graph.

The second part of the definition comes from [14]: it is proved that the whole number of tokens of any couple of backward places (p, p') of a WEG is constant and must be greater than or equal to $M_{min}(p)$ if the graph is live.

For our practical problem, we may considered a SWEG $\mathcal{G}' = (P', T, \ell)$ built from a WEG $\mathcal{G} = (P, T, \ell)$ by adding a backward place to any place $p \in P$. The liveness problem consists then to find an initial live marking $M(0, p)$, $p \in P'$. The places capacities are bounded and verify, for any $p \in P$ and its backward place p', $C(p) = M(0, p) + M(0, p')$. So, $M_{min}(p), p \in P$ is a lower bound of $C(p)$.

2.4 Unitary graphs and Normalization

2.4.1 Unitary graphs

Unitary WEG were introduced in [18] as follows:

Definition 2.1. *A unitary WEG $\mathcal{G}$ is strongly connected and such that every circuit of $\mathcal{G}$ has a unit weight.*

The SWEG tackled in this paper are unitary graphs: they are strongly connected. Moreover, if there exists a circuit c with $W(c) > 1$, then a circuit c' may be built by considering backward places with $W(c') < 1$. By the necessary condition of liveness expressed previously, we conclude that the SWEG is not live.

2.4.2 Normalization of a unitary graph

Normalization of a unitary marked WEG $\mathcal{G}$ is a transformation of all the marking functions and initial marking values such that the marking functions adjacent to any transition t_i have the same value Z_i. This transformation does not affect the constraints induced by places between the firings, so that these two graphs are equivalent. More formally:

Definition 2.2. *A transition t_i is normalized iff there exists $Z_i \in \mathbb{N}^\star$ such that:*

$$\begin{array}{lll} \forall p \in \mathcal{P}^+(t_i), & w(p) & = Z_i \textit{ and} \\ \forall p \in \mathcal{P}^-(t_i), & v(p) & = Z_i. \end{array}$$

A WEG G is said to be normalized iff all its transitions are normalized.

In [14], it is stated that any unitary Weighted Event Graph can be polynomially transformed into an equivalent normalized Weighted Event Graph by modifying marking functions and initial markings. So, we consider in this paper normalized SWEG.

2.4.3 Example

Let us consider the WTEG pictured by Figure 2. The corresponding SWTEG is pictured by Figure 3. Notice that it is a unitary graph.

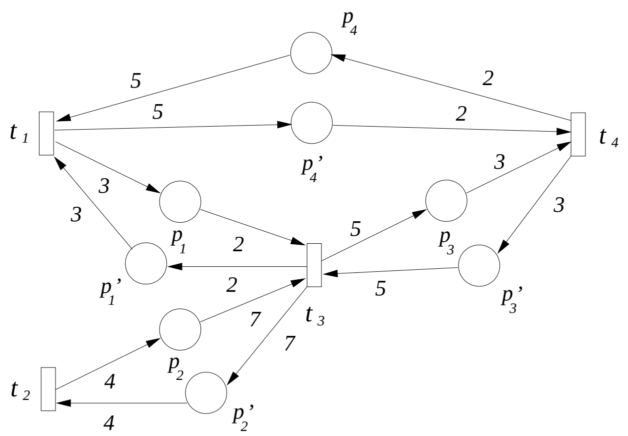

Figure 3: The Symmetric Timed Event Graph associated with the WTEG of Figure 2.

The normalization step consists in solving the following system:

$$\begin{array}{lll}\forall p \in P, & x(p) \in \mathbb{N}^\star & \\ \forall t_i \in T, & Z_i \in \mathbb{N}^\star & \\ \forall p \in \mathcal{P}^+(t_i), & w(p) \cdot x(p) & = Z_i \\ \forall p \in \mathcal{P}^-(t_i), & v(p) \cdot x(p) & = Z_i. \end{array}$$

It is proved in [14] that a solution exists. For our example, equations are $Z_1 = 3 \cdot x(p_1) = 5 \cdot x(p_4)$, $Z_3 = 7 \cdot x(p_2) = 2 \cdot x(p_1) = 5 \cdot x(p_3)$ and $Z_4 = 3 \cdot x(p_3) = 2 \cdot x(p_4)$. A solution is $x(p_1) = 35$, $x(p_2) = 10$, $x(p_3) = 14$, $x(p_4) = 21$ and $(Z_1, Z_2, Z_3, Z_4) = (105, 40, 70, 42)$. The corresponding normalized SWEG is pictured by Figure 4.

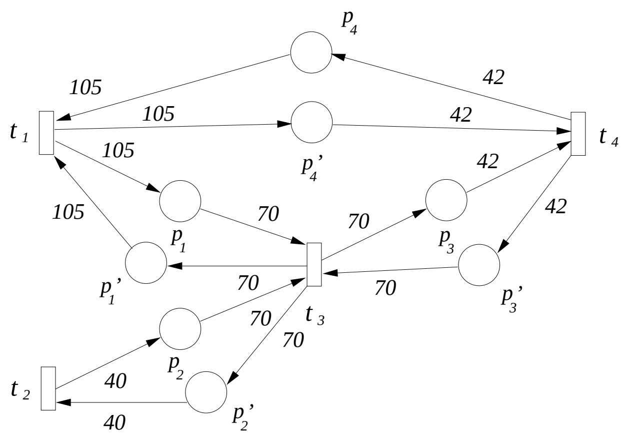

Figure 4: The Symmetric Normalized Timed Event Graph associated with the SWTEG of Figure 3.

2.5 Schedules and throughput of unitary normalized WTEGs

Let $\mathcal{G}$ be a unitary marked WTEG. A schedule is a function $\sigma : T \times \mathbb{N}^\star \rightarrow \mathbb{R}^{+\star}$ such that $\sigma(t_i, q)$ denotes the starting time of the q-th firing of t_i. There is a strong relationship between a schedule and the corresponding instantaneous marking. Let $p = (t_i, t_j)$ be a place of P. For any value $\tau \in \mathbb{R}^{+\star}$, let us denote by $E(\tau, t_i)$ the number of firings of t_i completed at time τ. More formally,

$$E(\tau, t_i) = \max\{q \in \mathbb{N}, \sigma(t_i, q) + \ell(t_i) \leq \tau\}.$$

On the same way, $B(\tau, t_j)$ denotes the number of firings of t_j at time τ and

$$B(\tau, t_j) = \max\{q \in \mathbb{N}, \sigma(t_j, q) \leq \tau\}.$$

Clearly,

$$M(\tau, p) = M(0, p) + w(p) \cdot E(\tau, t_i) - v(p) \cdot B(\tau, t_j).$$

A schedule (and its corresponding marking) is feasible if $M(\tau, p) \geq 0$ for every couple $(\tau, p) \in \mathbb{R}^{+\star} \times P$. The throughput of a transition t_i for a schedule σ is defined as

$$\lambda(t_i) = \lim_{q \to \infty} \frac{q}{\sigma(t_i, q)}.$$

Note that for live marked WTEG, earliest schedule (which consists on firing the transitions as soon as possible) always exists and has maximum throughputs. We consider here that the throughputs of a marked WTEG are those of its earliest schedule.

Definition 2.3. *A sequence u_n, $n \in \mathbb{N}^\star$ is K-periodic if there exists $(N_0, K, w) \in \mathbb{N}^2 \times \mathbb{R}^{+\star}$ such that, for any $n \geq N_0$, $u_{n+K} = u_n + w$. If $K = 1$, the sequence is said to be periodic.*

*$\frac{K}{w}$ is the throughput of the sequence and K its periodicity factor. A schedule σ is K-periodic (*resp.* periodic) if sequences $\sigma(t_i, k), k \in \mathbb{N}$ are K-periodic (*resp.* periodic), for any $t_i \in T$.*

For marked TEGs, setting the lenght of a circuit $\mathcal{C}$ to $L(\mathcal{C}) = \sum_{t \in T \cap \mathcal{C}} \ell(t)$, Chrétienne proved the following theorem:

Theorem 2.1. *[2] Let $\mathcal{G}$ be a strongly connected TEG with a live initial marking $M(\mathcal{G})$. The earliest schedule is then K-periodic and all the transitions have the same maximum throughput equal to*

$$\lambda(M(\mathcal{G})) = \min_{\mathcal{C} \in \mathcal{C}_\mathcal{G}} \frac{H(\mathcal{C})}{L(\mathcal{C})}$$

where $\mathcal{C}_\mathcal{G}$ denotes the set of circuits of $\mathcal{G}$.

A critical circuit c of $\mathcal{G}$ is such that $\frac{H(c)}{L(c)} = \lambda(M(\mathcal{G}))$. Using Koenig lemma (*see.* [1]), Chrétienne also proved the following theorem:

Theorem 2.2. *[2] Let $\mathcal{G}$ be a TEG with a live initial marking $M(\mathcal{G})$. Then, every critical circuit may be decomposed into elementary critical circuits.*

This last theorem allows us to consider only elementary critical circuits.

In the case of marked WTEG, the computation of the throughputs has an unknown complexity. Up to now, the algorithms developed are pseudo-polynomial time (*see.* as example [18]). However, we deduce from [18] and the definition of Z_i, $i \in \{1, \dots, n\}$ the following theorem:

Theorem 2.3. *Let $\mathcal{G}$ be a unitary WTEG with a live initial marking $M(\mathcal{G})$. Then, the earliest schedule is K-periodic and its throughputs verify, for any couple of transitions $(t_i, t_j) \in T^2$,*

$$\lambda(t_i) \cdot Z_i = \lambda(t_j) \cdot Z_j \stackrel{def}{=} \lambda(M(\mathcal{G})).$$

$\lambda(M(\mathcal{G}))$ is called the throughput of $\mathcal{G}$ associated with the initial marking $M(\mathcal{G})$.

Let us suppose now that the initial marking $M(\mathcal{G})$ of a strongly connected WTEG is such that, for the earliest schedule, at least one transition $t_{i^\star}$ is fired without any interruption. Then, $\lambda(t_{i^\star}) = \frac{1}{\ell(t_{i^\star})}$. By Theorem 2.3 and since $\lambda(t_i) \leq \frac{1}{\ell(t_i)}$ for any transition $t_i \in T$, we obtain:

$$\lambda(M(\mathcal{G})) = \frac{Z_{i^\star}}{\ell(t_{i^\star})} \leq \frac{Z_i}{\ell(t_i)}, \forall t_i \in T.$$

We deduce the following definition:

Definition 2.4. *The intrinsic throughput of a unitary WTEG $\mathcal{G}$ is defined as the ratio $\min_{t_i \in T} \left(\frac{Z_i}{\ell(t_i)}\right)$ and corresponds to the maximum throughput that may be achieved by an initial marking of $\mathcal{G}$.*

This value may always be achieved if we consider a sufficiently large number of tokens in each place.

Let us consider our previous example depicted by Figure 4. Then $\frac{Z_1}{\ell(t_1)} = \frac{105}{5} = 21$, $\frac{Z_2}{\ell(t_2)} = \frac{40}{2} = 20$, $\frac{Z_3}{\ell(t_3)} = \frac{70}{6} = \frac{35}{3}$ and $\frac{Z_4}{\ell(t_4)} = \frac{42}{3}$. So $t_{i^\star} = t_3$ and the intrinsic throughput of the SWTEG pictured by Figure 4 is $\frac{35}{3}$.

2.6 Problems formulations

Our purpose in this paper is to study the complexity of two sub-problems of the determination of a live marking of a WTEG maximizing the throughput while minimizing the places capacities. So, they both considered Symmetric Weighted Timed Event Graphs. The first one, MAX THROUGHPUT - MIN BOUNDED concerns the determination of a minimally bounded marking that maximizes the throughput for a STEG for unitary durations of firings. It is defined as follows:

MAX THROUGHPUT - MIN BOUNDED:

Instance: A STEG $\mathcal{G} = (T, P, \ell)$ such that $\ell(t) = 1, \forall t \in T$ and an integer $K > 2$.

Question: Is there an initial live marking $M(\mathcal{G})$ minimally bounded such that $\lambda(M(\mathcal{G})) \geqslant \frac{1}{K}$?

The complexity of MAX THROUGHPUT - MIN BOUNDED is studied in Section 3.

The second one concerns the determination for a SWTEG of an initial marking with intrinsic throughput and that uses a minimum number of initial tokens:

MAX INTRINSIC THROUGHPUT:

Instance: $\mathcal{G} = (P, T, \ell)$ is a normalized SWTEG, a vector y_p, $p \in P$ such that $y_p = y_{p'}$ for every couple (p, p') of backward places and $R \in \mathbb{N}^\star$.

Question: Is there an initial live marking $M(\mathcal{G})$ such that $\sum_{p \in P} y_p \cdot M(0, p) \leqslant R$ and $\lambda(M(\mathcal{G})) \geqslant \min_{t_i \in T} \left(\frac{Z_i}{\ell(t_i)} \right)$?

The complexity of MAX INTRINSIC THROUGHPUT, polynomial subcases and a 2-approximation algorithm are studied in Sections 4 and 5.

3 Complexity of Max Throughput - Min Bounded

The main result of this section is the NP-completness of MAX THROUGHPUT - MIN BOUNDED. We introduce first the problem FLOW RATIO as defined by Minty [17]. A relationship between FLOW RATIO and MAX THROUGHPUT - MIN BOUNDED is then exhibited. Their NP-COMPLETNESS is deduced using

Minty's lemma [17]. In the last part, we introduce the problem MARKING OPTIMIZATION which was studied by several authors [4, 12] due to the importance of the practical applications. We prove that it is NP-COMPLETE using the previous results.

3.1 Definition of Flow ratio

Let $G = (V, E)$ be a non-oriented simple graph. An edge orientation is a function $o : E \to V \times V$ such that, for any $e = \{x, y\} \in E$, $o(e) \in \{(x, y), (y, x)\}$.

The flow ratio $\rho(o(G))$ of an orientation o is then defined as follows: for any cycle c of G, we may associate two integers $n_c(o)$ and $m_c(o)$ corresponding respectively to the number of edges of c in a way and in its opposite. Then,

$$\rho(o(G)) = \max_{c \in \mathcal{C}_G} \left(\frac{n_c(o)}{m_c(o)}, \frac{m_c(o)}{n_c(o)} \right).$$

The associated decision problem is then defined as follows:
FLOW RATIO:

Instance: A non oriented graph $G = (V, E)$, an integer $L > 0$.

Question: Is there an orientation o of G such that $\rho(o(G)) \leq L$?

3.2 A transformation between Flow ratio and Max Throughput - Min Bounded

We define a transformation f from an instance of FLOW RATIO to an instance of MAX THROUGHPUT - MIN BOUNDED. Let I be an instance of FLOW RATIO given by a graph $G = (V, E)$ and an integer L. The corresponding instance $f(I)$ of MAX THROUGHPUT - MIN BOUNDED is defined as:

1. Any vertex $x \in V$ is associated with a transition $t_x \in T$ with $\ell(t_x) = 1$;
2. Any edge $e = \{x, y\} \in T$ corresponds to two places $p_1 = (t_x, t_y)$ and $p_2 = (t_y, t_x)$;
3. $K = L + 1$.

Now, if o is an orientation of G, a minimally bounded initial marking for $\mathcal{G}$ can be deduced by setting

1. $M(0,p) \in \{0,1\}$ for any $p \in P$;
2. For any place $p = (t_x, t_y)$, $M(0,p) = 1$ iff $o(\{x,y\}) = (x,y)$.

Notice that f is bijective. Moreover, any minimally bounded initial marking of $\mathcal{G}$ is associated with exactly one orientation of G. We are now ready to prove the relationship between $\rho(o(G))$ and $\lambda(M(\mathcal{G}))$.

Lemma 3.1. *For any orientation o of G associated with a minimally bounded initial marking M of $\mathcal{G}$,*

$$\lambda(M(\mathcal{G})) = \frac{1}{\rho(o(G)) + 1}.$$

Proof. Let c be a cycle of G. Then, c is associated with two circuits of $\mathcal{G}$ in opposite directions denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$ (*see.* Figure 5).

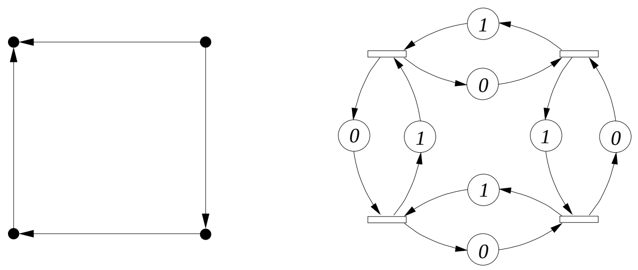

Figure 5: A cycle c and its orientation are depicted on the left hand side of the figure. The associated marked STEG is given on the right hand side.

We can suppose without loss of generality that the direction of $\mathcal{C}_1$ corresponds to the $n_c(o)$ arcs of c. So, $H(\mathcal{C}_1) = n_c(o)$ and $H(\mathcal{C}_2) = m_c(o)$. Moreover, $L(\mathcal{C}_1) = L(\mathcal{C}_2) = n_c(o) + m_c(o)$.

By Theorem 2.1, the throughput $\lambda_{\mathcal{C}}$ of the STEG composed only by $\mathcal{C}_1$ and $\mathcal{C}_2$ is:

$$\begin{aligned}
\lambda_{\mathcal{C}} &= \min\left(\tfrac{n_c(o)}{n_c(o)+m_c(o)}, \tfrac{m_c(o)}{n_c(o)+m_c(o)}\right) \\
&= \frac{1}{\max\left(\frac{n_c(o)+m_c(o)}{n_c(o)}, \frac{n_c(o)+m_c(o)}{m_c(o)}\right)} \\
&= \frac{1}{\max\left(\frac{m_c(o)}{n_c(o)}, \frac{n_c(o)}{m_c(o)}\right)+1}.
\end{aligned}$$

So, if ρ_c denotes the flow ratio of c, we get $\lambda_{\mathcal{C}} = \frac{1}{\rho_c+1}$. Now, if c is a cycle with a maximum flow ratio in G, one of the corresponding circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ have a minimum throughput in $\mathcal{G}$, so the equality is true. □

Lemma 3.2. *f is a polynomial transformation.*

Proof. f may be computed polynomially. The correctness of the transformation may be easily deduced from Lemma 3.1. □

3.3 Complexity of Flow ratio and Max Throughput - Min Bounded

Result described in [17] were established before the NP-COMPLETENESS theory advent. We clearly show that FLOW RATIO is intractable.

Theorem 3.1. FLOW RATIO *is* NP-COMPLETE.

Proof. FLOW RATIO belongs to NP since the flow ratio of an edge orientation may be computed efficiently from the throughput of the associated marked STEG (computed by Karp's algorithm [11] for example). Moreover, Minty's lemma [17] states that a graph $G = (V, E)$ is K-colorable if and only if there exists an edge orientation of E with a flow ratio lower than or equal to $K - 1$. This result can be seen as a polynomial reduction from GRAPH K-COLORABILITY to FLOW RATIO. As GRAPH K-COLORABILITY is NP-COMPLETE (*see.* [5]), the theorem holds. □

We deduce now our complexity result:

Theorem 3.2. MAX THROUGHPUT - MIN BOUNDED *is* NP-COMPLETE.

Proof. MAX THROUGHPUT - MIN BOUNDED belongs to NP since the throughput can be efficiently computed by using Karp's algorithm [11] and one can polynomially check that the marking properties are met.

Since f is bijective, f and f^{-1} define a polynomial reduction between FLOW RATIO and MAX THROUGHPUT - MIN BOUNDED. By Theorem 3.1, the proof is completed. In fact, these two problems are equivalent. □

3.4 Complexity of Marking Optimization

We derive from the last theorem the complexity of Marking Optimization problem. This last problem was considered by several authors because of the importance of practical applications [4, 12] but its complexity was up to now unknown.

Marking Optimization:

Instance: $\mathcal{G} = (P, T, \ell)$ is a TEG , Y is a P-semiflow of $\mathcal{G}$, $N \in \mathbb{N}^\star$ and $Q \in \mathbb{N}^\star$.

Question: Is there an initial marking $M(\mathcal{G})$ such that ${}^\intercal Y \cdot M(0,p) \leqslant N$ and $\lambda(M(\mathcal{G})) \geqslant \frac{1}{Q}$?

Theorem 3.3. Marking Optimization *is* NP-complete.

Proof. Marking Optimization is in NP. We prove that Max Throughput - Min Bounded is a sub-problem of Marking Optimization. Let I be an instance of Max Throughput - Min Bounded given by a STEG $\mathcal{G}$ and an integer K. Since $\mathcal{G}$ is a STEG, $|\mathcal{P}^+(t)| = |\mathcal{P}^-(t)|,\ \forall t \in T$. Hence, each column of the incidence matrix of the graph has exactly as many positive entries as negative ones. Therefore, the unit vector $1^{|P|}$ is a P-semiflow. Moreover, by setting $N = \frac{|P|}{2}$, the constraint ${}^\intercal Y \cdot M(0,p) \leqslant N$ is equivalent to require that M should be minimally bounded. Lastly, setting $Q = K$ we deduce that I is an instance of Marking Optimization. $\square$

4 Complexity of Max Intrinsic Throughput problem for a STEG

We study here the complexity of Max Intrinsic Throughput for a STEG. We first prove that it is NP-complete, even if the cost vector y is unit. We then show that it is polynomial if the durations of the transitions are unit and we derive a 2-approximation solution in the general case.

4.1 Complexity

Theorem 4.1. Max Intrinsic Throughput *is* NP-complete *for a STEG with $y_p = 1,\ \forall p \in P$.*

Proof. The problem belongs to NP. We exhibit a reduction from Max Throughput - Min Bounded to Max Intrinsic Throughput. Let an instance of Max Throughput - Min Bounded defined by a STEG $\mathcal{G} = (P, T, \ell)$ and an integer value K. We build a corresponding instance of Max Intrinsic Throughput as follows:

1. A STEG $\mathcal{G}^\star$ is built by adding a transition $t^\star$ with $\ell(t^\star) = K$ and a couple of backward places $p = (t, t^\star)$ and $p' = (t^\star, t)$ for a fixed transition $t \in T$.

2. We set $R^\star = \frac{|P|}{2} + 2$. Since $\mathcal{G}$ is an event graph, we have $Z_i = 1$ for all transitions $t_i \in T$. Therefore, its intrinsic throughput is equal to $\min\limits_{t_i \in T}\left(\frac{Z_i}{\ell(t_i)}\right) = \frac{1}{\max\limits_{t_i \in T}(\ell(t_i))} = \frac{1}{K}$.

- If M is a solution for the instance of Max Throughput - Min Bounded, we build a solution $M^\star$ for the corresponding instance of Max Intrinsic Throughput by adding 1 token on each added places. The total number of initial tokens is $R^\star = \frac{|P|}{2} + 2$. Now, if $c^\star$ is a critical circuit of $\mathcal{G}^\star$, then by Theorem 2.2, $c^\star$ can be considered as elementary. We get then two cases:

  1. If $c^\star$ is included in $\mathcal{G}$, then $\lambda(M^\star(\mathcal{G}^\star)) \geq \frac{H(c^\star)}{L(c^\star)} \geq \frac{1}{K}$;
  2. Else, since $c^\star$ is elementary, it composed only by the two transitions t and $t^\star$ and thus
  $$\frac{H(c^\star)}{L(c^\star)} = \frac{2}{K+1} > \frac{1}{K}.$$

  So, $M^\star$ is a solution to the corresponding instance of Max Intrinsic Throughput.

- Let us suppose now that $M^\star$ is a solution for the instance of Max Intrinsic Throughput. Setting $M(0, p) = M^\star(0, p)$ for every place $p \in P$, we prove that M is a solution to the corresponding instance of Max Throughput - Min Bounded.

  By definition of M and $M^\star$, $\lambda(M(\mathcal{G})) \geq \lambda(M^\star(\mathcal{G}^\star)) \geq \frac{1}{K}$.

  Now, let c be the circuit of $\mathcal{G}^\star$ composed by transitions t and $t^\star$. Since $M^\star$ is a live marking, $H(c) \geq 1$. So, $\frac{H(c)}{L(c)} = \frac{H(c)}{K+1} \geq \frac{1}{K}$. So, $H(c) \geq 1 + \frac{1}{K}$ and then c has at least two tokens.

So, the number of initial tokens of $\mathcal{G}$ is upper bounded by $\frac{|P|}{2}$. Since $M^\star$ is live, M is live and then, for every couple (p, p') of backward places,

$$M(0, p) + M(0, p') \geq 1.$$

So, there is exactly one token for every couple of backward places and M is minimally bounded.

□

4.2 A polynomial special case and an approximation algorithm

Theorem 4.2. Max Intrinsic Throughput *is polynomial for a STEG $\mathcal{G} = (P, T, \ell)$ with $\ell(t) = 1$, $\forall t \in T$.*

Proof. We prove that the initial marking $M(0, p) = 1$, $\forall p \in P$ is an optimal solution.

This marking is live. Moreover, since $\ell(t_i) = Z_i = 1, \quad \forall t_i \in T$, then

$$\min_{t_i \in T} \left(\frac{Z_i}{\ell(t_i)} \right) = \frac{1}{\max\limits_{t_j \in T} (\ell(t_j))} = 1.$$

Therefore, we have $\lambda(M(\mathcal{G})) = 1$. So, the initial marking is feasible.

We prove now that it minimizes the criteria $\sum\limits_{p \in P} y_p \cdot M(0, p)$: as $\mathcal{G}$ is symmetric, every couple of backward places needs at least two tokens to reach the throughput 1. So, $\sum\limits_{p \in P} y_p$ is a lower bound of a solution, the result. □

If the durations of the firings are not required to be equal to 1, the previous algorithm builds an initial marking with an intrinsic maximum throughput. The total number of tokens is not necessarily minimum, as illustrated by Figure 6.

Corollary 4.1. *Setting $M(0, p) = 1$, $\forall p \in P$ leads to a 2-approximation solution for the* Max Intrinsic Throughput *problem.*

Proof. The marking is live. Moreover, for every circuit c with q transitions, we get $\frac{H(c)}{L(c)} \geq \frac{q}{q \cdot \max\limits_{t_i \in T}(\ell(t_i))}$. So, $\lambda(M(\mathcal{G})) = \frac{1}{\max\limits_{t_i \in T}(\ell(t_i))}$ and the marking is a

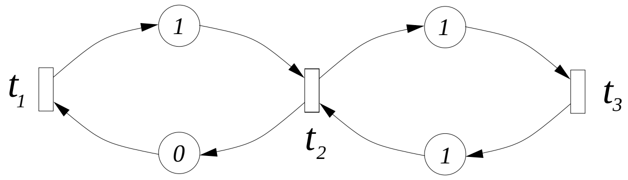

Figure 6: The STEG has three transitions with $\ell(t_1) = 1$, $\ell(t_2) = 2$ and $\ell(t_3) = 3$. The initial marking $M^\star$ reaches the maximum intrinsic throughput of the system *i.e.* $\frac{1}{3}$.

feasible solution. Now, every couple of backward places needs at least one tokens for checking the liveness condition. So, $\sum_{p \in P} \frac{1}{2} \cdot y_p$ is an upper bound on the criteria and $\sum_{p \in P} y_p \cdot M(0,p) \leq 2 \sum_{p \in P} \frac{1}{2} \cdot y_p$. □

Notice that this bound is asymptotically tight. Indeed, if we set, for the example of Figure 6, $y_p = r \in \mathbb{N}^\star$ for $p \in \{(t_1, t_2), (t_2, t_1)\}$ and $y_p = 1$ for $p \in \{(t_2, t_3), (t_3, t_2)\}$, we get by the algorithm $\sum_{p \in P} y_p \cdot M(0,p) = 2 \cdot r + 2$. Now, for the marking $M^\star$ presented by Figure 6, $\sum_{p \in P} y_p \cdot M^\star(0,p) = r + 2$ and we get the ratio 2 when r tends to infinity.

These two previous results are extended to SWTEG in next section.

5 A polynomial special case of Max Intrinsic Throughput problem for a SWTEG

We first consider a couple of backward places (p_1, p_2) between two transitions t_i and t_j with $p_1 = (t_i, t_j)$, $v(p_1) = w(p_2) = Z_j$, $v(p_2) = w(p_1) = Z_i$ and $\frac{Z_i}{\ell(t_i)} = \frac{Z_j}{\ell(t_j)} = \rho$. This SWTEG is denoted by $\mathcal{G}_{p_1,p_2}$ (*see.* Figure 7). We prove that the initial marking $M(0,p_1) = M(0,p_2) = M_{min}(p_1)$ is an optimal solution for the decision problem associated with Max Intrinsic Throughput problem for this graph. The extension of the results of the previous section to SWTEG is then deduced.

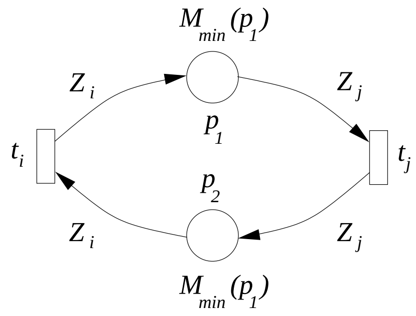

Figure 7: A SWTEG $\mathcal{G}_{p_1,p_2}$ marked with $M(0,p_1) = M(0,p_2) = M_{min}(p_1)$.

5.1 Feasibility of the initial marking for $\mathcal{G}_{p_1,p_2}$

Lemma 5.1. *The schedule $\sigma(t,k) = (k-1)\cdot \ell(t)$ for $t \in \{t_i, t_j\}$ is feasible for the initial marking $M(0,p_1) = M(0,p_2) = M_{min}(p_1)$ and its throughput equals the intrinsic throughput of $\mathcal{G}_{p_1,p_2}$.*

Proof. By definition of σ, for any $p = (t,t')$, $t \neq t'$ with $(t,t') \in \{t_i,t_j\}^2$ and $\tau \in \mathbb{R}$, the number of firings of t completed at time τ is $E(\tau,t) = \left\lfloor \frac{\tau}{\ell(t)} \right\rfloor = \left\lfloor \frac{\tau}{\rho\cdot w(p)} \right\rfloor$. In the same way, the number of firings of t' at time τ is $B(\tau,t') = \left\lceil \frac{\tau}{\ell(t')} \right\rceil = \left\lceil \frac{\tau}{\rho\cdot v(p)} \right\rceil$. So, the instantaneaous marking of p can be described by the following equations:

$$\left\{ \begin{array}{lcl} M(0,p) & = & w(p) + v(p) - gcd_p \\ M(\tau,p) & = & M(0,p) + \left\lfloor \frac{\tau}{\rho\cdot w(p)} \right\rfloor \cdot w(p) - \left\lceil \frac{\tau}{\rho\cdot v(p)} \right\rceil \cdot v(p) \quad \forall \tau > 0 \end{array} \right.$$

We prove that $M(\tau,p) \geq 0$ for any value $\tau \geq 0$ by contradiction. Assume that there exists a value $\tau \geq 0$ such that $M(\tau,p) < 0$. Since $M(\tau,p)$ is a linear combination of $w(p)$ and $v(p)$ it follows that $M(\tau,p) \equiv 0 \mod gcd_p$ and therefore

$$M(\tau,p) < 0 \Longleftrightarrow M(\tau,p) \leqslant -gcd_p.$$

So, by definition of $M(\tau,p)$, we obtain

$$M(0,p) + \left\lfloor \frac{\tau}{\rho\cdot w(p)} \right\rfloor \cdot w(p) - \left\lceil \frac{\tau}{\rho\cdot v(p)} \right\rceil \cdot v(p) \leqslant -gcd_p.$$

Since $M(0,p) = M_{min}(p) = w(p) + v(p) - gcd_p$, we deduce that

$$w(p) + v(p) + \left\lfloor \frac{\tau}{\rho \cdot w(p)} \right\rfloor \cdot w(p) \leqslant \left\lceil \frac{\tau}{\rho \cdot v(p)} \right\rceil \cdot v(p)$$

However $\forall x \in \mathbb{R}$ we have

$$x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1.$$

Therefore

$$w(p) + v(p) + \frac{\tau}{\rho} - w(p) < \frac{\tau}{\rho} + v(p)$$

and then

$$0 < 0.$$

A contradiction. So, the schedule is feasible. Moreover, we have $\lambda(t) = \frac{1}{\ell(t)}$ for $t \in \{t_i, t_j\}$. As $\rho = \frac{Z_i}{\ell(t_i)} = \frac{Z_j}{\ell(t_j)}$, we deduce by Theorem 2.3 that the throughput of σ equals ρ and is exactly the intrinsic throughput of $\mathcal{G}_{p_1,p_2}$. $\square$

5.2 Earliest schedule of $\mathcal{G}_{p_1,p_2}$

We suppose here that $\mathcal{G}_{p_1,p_2}$ has an initial marking $M(\mathcal{G})$ which values are not necessarily equal to $M_{min}(p_1)$.

Lemma 5.2. *If the throughput of the earliest schedule s of $\mathcal{G}_{p_1,p_2}$ equals ρ for a given marking, then s is periodic. Moreover, it exists $\tau^\star \in \mathbb{R}^+$ such that, for any $t \in \{t_i, t_j\}$ there exists $k_t \in \mathbb{N}$ such that $s(t,k) = \tau^\star + (k - k_t) \cdot \ell(t)$.*

Proof. By Theorem 2.3, the earliest schedule s is K-periodic. Now, since the system throughput is intrinsically maximum, there is no idle times when time τ tends to infinity and the periodicity factor K of the sequences is unit.

So, for any $t \in \{t_i, t_j\}$, there exists a couple $(\tau^\star(t), k(t)) \in \mathbb{R} \times \mathbb{N}^\star$ with minimum values such that, for any $k \geq k(t)$, $s(t,k) = \tau^\star(t) + (k - k(t))\ell(t)$.

If $\tau^\star(t_j) = \tau^\star(t_i)$, the lemma is proved by setting $\tau^\star = \tau^\star(t_j)$ and $k_t = k(t)$ for $t \in \{t_i, t_j\}$.

Let us suppose now that $\tau^\star(t_j) > \tau^\star(t_i)$. Then, by minimality of $\tau^\star(t_j)$, there is an idle slot before $s(t_j, k(t_j))$ due to a precedence constraint before the $k(t_j)$ firing of t_j. So, $s(t_j, k(t_j))$ is the completion time of a firing of t_i and there exists an integer k' such that $s(t_i, k') = s(t_j, k(t_j))$. Setting $\tau^\star = \tau^\star(t_j)$, $k_{t_i} = k'$ and $k_{t_j} = k(t_j)$, we obtain the result.

Lastly, if $\tau^\star(t_i) > \tau^\star(t_j)$, the same reasoning holds by swapping i and j. $\square$

In order to show that our solution is minimum, we have to study the marking of places of $\mathcal{G}_{p_1,p_2}$ whenever both transitions t_i and t_j are executed at the same instant time. In this purpose, we define a special period of time.

Let us now define N_i and N_j as the smallest strictly positive integers such that $N_i \cdot w(p_1) = N_j \cdot v(p_1)$. Observe that, since $w(p_1) = Z_i$ and $v(p_1) = Z_j$, $N_i \cdot Z_i = N_j \cdot Z_j$ and $\frac{\ell(t_i)}{\ell(t_j)} = \frac{Z_i}{Z_j} = \frac{N_j}{N_i}$. We denote $H = \ell(t_i) \cdot N_i = \ell(t_j) \cdot N_j$ the hyperperiod of the graph $\mathcal{G}_{p_1,p_2}$. This value is associated with the earliest schedule of $\mathcal{G}_{p_1,p_2}$. Therefore, by lemma 5.2, for each date $\tau \geqslant \tau^\star$ the instantaneous marking $M(\tau, p)$ of place $p \in \{p_1, p_2\}$ is defined by:

$$M(\tau,p) = M(\tau^\star,p) + \left\lfloor \frac{\tau-\tau^\star}{\rho \cdot w(p)} \right\rfloor \cdot w(p) - \left\lceil \frac{\tau-\tau^\star}{\rho \cdot v(p)} \right\rceil \cdot v(p)$$

Then, for each date $\tau \geqslant \tau^\star$, we have:

$$\begin{array}{rcl} M(\tau+H,p) & = & M(\tau^\star,p) + \left\lfloor \frac{\tau+H-\tau^\star}{l(t_i)} \right\rfloor \cdot w(p) - \left\lceil \frac{\tau+H-\tau^\star}{l(t_j)} \right\rceil \cdot v(p) \\ & = & M(\tau^\star,p) + \left(\left\lfloor \frac{\tau-\tau^\star}{l(t_i)} \right\rfloor + N_i \right) \cdot w(p) - \left(\left\lceil \frac{\tau-\tau^\star}{l(t_j)} \right\rceil + N_j \right) \cdot v(p) \\ & = & M(\tau^\star,p) + \left\lfloor \frac{\tau-\tau^\star}{l(t_i)} \right\rfloor \cdot w(p) - \left\lceil \frac{\tau-\tau^\star}{l(t_j)} \right\rceil \cdot v(p) \\ & = & M(\tau,p) \end{array}$$

So, we can reduce the study of marking to a period of time that lasts an hyperperiod H.

Now, observe that the time interval $[\tau^\star, \tau^\star + H[$ contains exactly N_i firings of t_i and N_j firings of t_j such that:

- t_i and t_j are fired simultaneously at time $\tau^\star$,
- by minimality of N_i and N_j, there are exactly $N_i - 1$ firings of t_i and $N_j - 1$ firing of t_j in the interval $]\tau^\star, \tau^\star + H[$ and they cannot be simultaneously (otherwise N_i and N_j would not be minimum).

So, a strictly increasing sequence of real number $s_h, h \in \{0, \ldots, N_i + N_j - 1\}$ can be defined such that $s_0 = \tau^\star$, $s_{N_i+N_j-1} = \tau^\star + H$ and, for any $r \in \{1, \ldots, N_i + N_j - 2\}$, a firing of t_i or t_j occurs at time s_r. We also define the interval sequence C_r, $r \in \{1, \ldots, N_i + N_j - 1\}$ as $C_r =]s_{r-1}, s_r[$.

For example, let us consider the couple of backwards places pictured by Figure 8. We assume that the initial marking (not presented in Figure) can not prevent any firing of transition t_i and t_j. We have $N_i = 5$ and $N_i = 2$. Then $H = 20$ and a slice of the earliest schedule between $\tau^\star$ and $\tau^\star + H$ is pictured by Figure 9. We get $s_0 = \tau^\star$, $s_1 = \tau^\star + 4$, $s_2 = \tau^\star + 8$, $s_3 = \tau^\star + 10$, $s_4 = \tau^\star + 12$, $s_5 = \tau^\star + 16$ and $s_6 = \tau^\star + 20$.

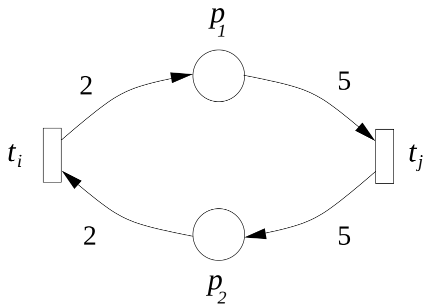

Figure 8: A couple of backward place with $\ell(t_i) = 4$ and $\ell(t_j) = 10$. Notice that $\frac{\ell(t_i)}{Z_i} = \frac{\ell(t_j)}{Z_j} = 2$.

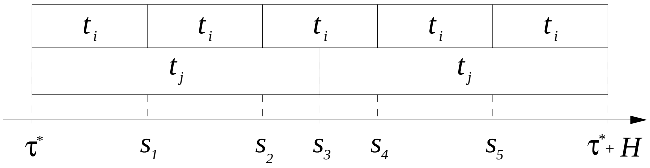

Figure 9: The earliest schedule between $\tau^\star$ and $\tau^\star + H$.

Now, for any $r \in \{1, \ldots, N_i + N_j - 2\}$, u_r denotes the transition of $\mathcal{G}_{p_1,p_2}$ fired at time s_r. $u = u_1 \ldots u_{N_i+N_j-2}$ is then the firing sequence of $\mathcal{G}_{p_1,p_2}$ during the time interval $]\tau^\star, \tau^\star + H[$.

For our last example, we get $u_1 = t_i$, $u_2 = t_i$, $u_3 = t_j$, $u_4 = t_i$ and $u_5 = t_i$.

We build now a marked Weighted Event Graph (not timed) denoted by $\mathcal{G}'_{p_1,p_2}$ as follows:

1. $\mathcal{G}'_{p_1,p_2}$ has the same structure as $\mathcal{G}_{p_1,p_2}$ (same places and same transitions).

2. Its initial marking is $M'_0(p) = M(\tau^\star, p) - v(p)$ for $p \in \{p_1, p_2\}$.

Lemma 5.3. *$u = u_1 \dots u_{N_i+N_j-2}$ is a firing sequence of ${\mathcal{G}'}_{p_1,p_2}$.*

Proof. Notice that, for any couple $(\tau_1, \tau_2) \in C_r^2$, $r \in \{1, \dots, N_i + N_j - 1\}$, $M(\tau_1, p) = M(\tau_2, p)$ for $p \in \{p_1, p_2\}$. So, the marking function M of $\mathcal{G}_{p_1,p_2}$ is constant on C_r, $r \in \{1, \dots, N_i + N_j - 1\}$.

For any $r \in \{1, \dots, N_i + N_j - 1\}$, we prove by recurrence that the marking of places of ${\mathcal{G}'}_{p_1,p_2}$ after the sequence of firings $u_1 \dots u_{r-1}$ is exactly $M(\tau, p)$, $p \in \{p_1, p_2\}$ and $\tau \in C_r$.

1. For any $\tau \in C_1$, $M(\tau, p_1) = M(\tau^\star, p_1) - v(p_1)$ and $M(\tau, p_2) = M(\tau^\star, p_2) - v(p_2)$. So it is true for $r = 1$.

2. Now, let us suppose that it is true for a value $1 \leq r < N_i + N_j - 1$. Let us consider a couple of times $(\tau_1, \tau_2) \in C_r \times C_{r+1}$.

   - If $u_r = t_i$, then we get $M(\tau_2, p_1) = M(\tau_1, p_1) + w(p_1)$ and $M(\tau_2, p_2) = M(\tau_1, p_2) - w(p_1)$. By hypothesis, the markings of p_1 and p_2 after the sequence $u_1 \dots u_{r-1}$ are respectively equal to $M'(p_1) = M(\tau_1, p_1)$ and $M'(p_2) = M(\tau_1, p_2)$. So, after the firing of $u_r = t_i$, they are respectively equal to $M(\tau_2, p_1)$ and $M(\tau_2, p_2)$.
   - In the same way, if $u_r = t_j$, we obtain $M(\tau_2, p_1) = M(\tau_1, p_1) - w(p_2)$ and $M(\tau_2, p_2) = M(\tau_1, p_2) + w(p_2)$. The property is also true in this case.

So, we conclude that $u = u_1 \dots u_{N_i+N_j-2}$ is a firing sequence of ${\mathcal{G}'}_{p_1,p_2}$. ☐

A state of $\mathcal{G}'_{p_1,p_2}$ is given by a couple of marking $\begin{pmatrix} M'(p_1) \\ M'(p_2) \end{pmatrix}$. Next subsection provides some properties on the number of different states of ${\mathcal{G}'}_{p_1,p_2}$.

5.3 Upper bound on the number of states of $\mathcal{G}'_{p_1,p_2}$

Lemma 5.4. *If $M'_0(p_1) + M'_0(p_2) < v(p_1) + w(p_1) - 2 \cdot gcd_{p_1}$ then all the states of ${\mathcal{G}'}_{p_1,p_2}$ are distinct.*

Proof. According to [14], the marked WEG ${\mathcal{G}'}_{p_1,p_2}$ is live iff $M'_0(p_1) + M'_0(p_2) \geq v(p_1) + w(p_1) - gcd_{p_1}$. So, it is not live and all the intermediary states are different (otherwise, the system would be live). ☐

Lemma 5.5. *If $M'_0(p_1) + M'_0(p_2) < v(p_1) + w(p_1) - 2 \cdot gcd_{p_1}$ then the number of states of ${\mathcal{G}'}_{p_1,p_2}$ is at most $N_i + N_j - 2$.*

Proof. By [14], we know that the number of tokens hold in a place p is a multiple of gcd_p and that the number of tokens present in both places is constant for all valid firing sequences. It follows that we can consider that $M'_0(p_1) + M'_0(p_2) \leq w(p_1) + v(p_1) - 3 \cdot gcd_{p_1}$. Setting $S = v(p_1) + w(p_1) - 3 \cdot gcd_{p_1}$, the different possible states $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{N}^2$ can be enumerated under the following conditions:

$$\begin{cases} a & \equiv & 0 \mod (gcd_{p_1}) \\ b & \equiv & 0 \mod (gcd_{p_1}) \\ a+b & = & S \end{cases}$$

which yields the following enumeration

$$\underbrace{\begin{array}{c|c|c|c|c|c} 0 & gcd_{p_1} & 2 \cdot gcd_{p_1} & \dots & S - gcd_{p_1} & S \\ & & & & & \\ S & S - gcd_{p_1} & S - 2 \cdot gcd_{p_1} & \dots & gcd_{p_1} & 0 \end{array}}_{\text{Couples } \begin{pmatrix} a \\ b \end{pmatrix} \text{ with } a+b=S}$$

$$\text{This yields to } \frac{S}{gcd_{p_1}} + 1 \text{ couples}$$

By definition of N_i and N_j, we also have $N_i \cdot w(p_1) = N_j \cdot v(p_1) = lcm_{p_1}$. Since $w(p_1) \cdot v(p_1) = lcm_{p_1} \cdot gcd_{p_1}$,

$$\frac{w(p_1)}{gcd_{p_1}} = N_j \quad \text{and} \quad \frac{v(p_1)}{gcd_{p_1}} = N_i.$$

So the number of different states is equal to $N_i + N_j - 2$ for S initial tokens and lemma holds. ☐

5.4 Polynomial special cases and an approximation algorithm

We are now ready to show our main result:

Theorem 5.1. *Let $\mathcal{G}_{p_1,p_2}$ be a two backward places system such that*

$$\forall t_i \in T, \quad \ell(t_i) = \rho \cdot Z_i \ , \quad \rho > 0.$$

Then setting $M(0,p_1) = M(0,p_2) = M_{min}(p_1)$, we obtain a system with an intrinsic maximum throughput and such that the sum of initial markings $M(0,p_1) + M(0,p_2)$ is minimum.

Proof. By Lemma 5.1, the maximum throughput of the system equals the intrinsic throughput. So, we must prove that $M(0,p_1)+M(0,p_2)$ is minimum to achieve the throughput ρ.

By contradiction, let us suppose that there exists a marking M such that $M(0,p_1) + M(0,p_2) < 2 \cdot M_{min}(p_1)$ with a throughput equal to the intrinsic maximum value. According to lemma 5.2 page 21, there is a date $\tau^\star$ such that transitions t_i and t_j are both fired. Therefore, without loss of generality, we can consider that the marking M is such that $\tau^\star = 0$. Then, the initial marking of the corresponding WEG $\mathcal{G}'_{p_1,p_2}$ is well defined and verifies $M'_0(p_1) + M'_0(p_2) < v(p_1) + w(p_1) - 2 \cdot gcd_{p_1}$. So, by Lemma 5.5, the number of intermediary states is at most $N_i + N_j - 2$. But, by Lemmas 5.3 and 5.4, the number of intermediary states of $\mathcal{G}'_{p_1,p_2}$ is $N_i + N_j - 1$, the contradiction. □

Applying this last theorem to every couple of backward places of a SWTEG, we get the following corollary:

Corollary 5.1. *Let $\mathcal{G}$ be a unitary SWEGT such that*

$$\forall t_i \in T, \quad \ell(t_i) = \rho \cdot Z_i \ , \quad \rho > 0.$$

Then setting $M(0,p) = M_{min}(p)$ for every place $p \in P$, we obtain a system with an intrinsic maximum throughput and such that $\sum_{p \in P} y_p \cdot M(0,p)$ is minimum.

We deduce an approximation algorithm for the general case:

Theorem 5.2. *Let $\mathcal{G}$ be a unitary SWTEG. Setting $M(0,p) = v(p)+w(p)-gcd_p$ $\forall p \in P$ leads to a 2-approximation solution for the* Max Intrinsic Throughput *problem.*

Proof. Let $i^\star \in \{1,\ldots,n\}$ such that $\frac{Z_{i^\star}}{\ell(t_{i^\star})} = \min_{t_i \in T} \left(\frac{Z_i}{\ell(t_i)}\right)$. We set $\rho = \frac{Z_{i^\star}}{\ell(t_{i^\star})}$.

We first prove that the solution is live and that its throughput is equal to ρ. We show then that the criteria is at most twice from a minimum value.

1. Let us consider the periodic schedule $\sigma(t,k) = (k-1)\cdot\frac{Z_t}{\rho}$ for any $(t,k)\in T\times\mathbb{N}^\star$. Its troughput is ρ. Moreover, if we replace the duration of the firing sequences by values $\ell'(t_j)=\frac{Z_j}{\rho}\geq \ell(t_j)$, then the schedule σ is feasible for M by Lemma 5.1. So, we conclude that σ is feasible and that the system is live.

2. By [14], every live marking verify, for every couple (p,p') of backward places,
$$M(0,p)+M(0,p')\geq M_{min}(p).$$
So, $\frac{1}{2}\sum\limits_{p\in P} y_p\cdot M_{min}(p)$ is a lower bound of the criteria and the ratio is obtained.

□

For the example pictured by Figure 4, we obtain the initial marking $M(0,p_1)=M(0,p_1')=140$, $M(0,p_1)=M(0,p_1')=100$, $M(0,p_1)=M(0,p_1')=98$, $M(0,p_1)=M(0,p_1')=126$.

Our purpose was initially to bound the capacity storage of places of Figure 2. We obtain then $C(p_1)=\frac{M(0,p_1)+M(0,p_1')}{x(p_1)}=\frac{280}{35}=8$, $C(p_2)=\frac{M(0,p_2)+M(0,p_2')}{x(p_2)}=\frac{200}{10}=20$, $C(p_3)=\frac{M(0,p_3)+M(0,p_3')}{x(p_3)}=\frac{196}{14}=14$, and $C(p_4)=\frac{M(0,p_4)+M(0,p_4')}{x(p_4)}=\frac{252}{21}=12$. The global surface on the chip is then equal to $8\cdot y_{p_1}+20\cdot y_{p_2}+14\cdot y_{p_3}+12\cdot y_{p_4}$ and is less than twice from the optimum value. The restriction of M defines a live marking with throughput $\rho=\frac{35}{3}$.

6 Conclusions

We have presented in this paper several complexity results and approximation results on a bi-criteria cyclic scheduling problems.

We firstly have exhibited an equivalence between MAX THROUGHPUT - MIN BOUNDED problem and a K-colorability problem. The NP-completness of several cyclic bi-criteria scheduling problem was deduced. In the future, we hope that this equivalence should lead to original algorithms for its resolution.

We also developed a very simple approximation algorithm for MAX INTRINSIC THROUGHPUT problem. It will allow to solve efficiently this problem and to get good lower bounds to evaluate the performance of other algorithms.

The complexity of the general bi-criteria problem is still an interesting open question: up to now, there is no polynomial time algorithm to compute the liveness and the throughput of a Weighted Timed Event Graph. We do not know if it belong to NP and therefore devising efficient heuristics for such problem is a challenge.

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